Lezione n. 10

Formulazioni integrali (formulazione integrale di volume per problemi di correnti indotte, cenni su boundary elements)

Field equations

The mathematical model describing the eddy current problem in non-magnetic conductors at low frequencies is given by [1] the quasi-stationary Maxwell equations:

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{A.1.1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{A.1.2}$$

in the conducting domain V_c, with the constitutive properties:

$$\mathbf{J} = \sigma \mathbf{E} \tag{A.1.3}$$

$$\mathbf{B} = \mu_0 \mathbf{H} \tag{A.1.4}$$

and suitable initial conditions giving $\nabla \cdot \mathbf{B} = 0$ everywhere and $\|\mathbf{B}\| \cdot \hat{\mathbf{n}} = 0$ at any material interface.

Here $\hat{\mathbf{n}}$ is the normal to the interface and $\|\cdot\|$ stands for the jump of the quantity at the interface.

In the external region $\Re^3 - V_c$ the field equations are

$$\nabla \times \mathbf{H} = \mathbf{J}_s \tag{A.1.5}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{A.1.6}$$

with

$$\mathbf{B} = \mu_0 \mathbf{H} \tag{A.1.7}$$

The problem is closed by regularity conditions at the infinity and the interface conditions:

$$\|\mathbf{H}\| \times \hat{\mathbf{n}} = \mathbf{0} \tag{A.1.8}$$

$$\left\| \frac{\partial \mathbf{B}}{\partial t} \right\| \cdot \hat{\mathbf{n}} = 0 \tag{A.1.9}$$

Magnetic vector potential

Faraday's law (A.1.2) is automatically satisfied by assuming

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi \tag{A.1.10}$$

The magnetic vector potential is defined by $\nabla \times \mathbf{A} = \mathbf{B}$ with the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ and regularity conditions at infinity. Consequently, eqs.(A.1.2), (A.1.4) and (A.1.5) allow to express $\mathbf{A}$ in terms of the unknown solenoidal current density:

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_{V_c} \frac{\mathbf{J}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} dV' + \mathbf{A}_s(\mathbf{r}, t) \tag{A.1.11}$$

where $\mathbf{A}_s$ is the magnetic vector potential given by the sources located in $\Re^3 - V_c$:

$$\mathbf{A}_s(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_{\Re^3 - V_c} \frac{\mathbf{J}_s(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} dV' \tag{A.1.12}$$

Weighted residuals

Constitutive equation (A.1.3) can be verified in an average sense by adopting the weighted residual approach:

$$\int_{V_c} \mathbf{W} \cdot (\sigma^{-1}\mathbf{J} - \mathbf{E})\, dV = 0 \qquad \forall \mathbf{W} \in S, \tag{A.1.13}$$

where S is the subspace of $\mathbf{L}^2_{div}(V_c)$ defined by

$$S = \{\mathbf{J} \in \mathbf{L}^2_{div}, \nabla \cdot \mathbf{J} = 0 \,\mathrm{in}\, V_c, \mathbf{J} \cdot \hat{\mathbf{n}} = 0 \text{ on} \,\partial V_c\}. \tag{A.1.14}$$

Substituting (A.1.10) and (A.1.11) in (A.1.12), we finally obtain the weak form

$$\int_{V_c} \mathbf{W} \cdot \left\{ \sigma^{-1}\mathbf{J} + \frac{\partial}{\partial t}\left[\frac{\mu_0}{4\pi} \int_{V_c} \frac{\mathbf{J}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} dV' + (\mathbf{r}, t) \right] + \frac{\partial \mathbf{A}_s}{\partial t} \right\} dV = 0, \quad \forall \mathbf{W} \in S \tag{A.1.15}$$

$$\mathbf{J} \in S \tag{A.1.16}$$

In (A.1.15) the term related to the scalar potential ϕ disappeared because of the solenoidality of the weighting functions $\mathbf{W}$ and the boundary conditions $\mathbf{W} \cdot \hat{\mathbf{n}} = 0$ on ∂V_c. The possible presence of electric terminals will be considered in section 5.

Finite element formulation

A numerical solution is obtained by applying Galerkin's method to eq. (A.1.15). The unknown is the current density vector $\mathbf{J}$ which we represent as the linear combination of n basis functions $\mathbf{J}_k \in S$:

$$\mathbf{J}(\mathbf{r},t) = \sum_e I_k(t)\mathbf{J}_k(\mathbf{r}) \qquad \text{(A.2.1)}$$

According to the Galerkin's method we choose n independent weighting functions as $\mathbf{W}_k = \mathbf{J}_k$.

Condition $\mathbf{J}_k \in S$ can be satisfied by introducing the electric vector potential $\mathbf{T}$ ($\mathbf{J} = \nabla \times \mathbf{T}$) and adopting edge element shape functions (see Section 3) for $\mathbf{T}$. The uniqueness of the vector potential is assured by the gauge $\mathbf{T} \cdot \mathbf{w} = 0$ where $\mathbf{w}$ is an arbitrary vector field which does not possess closed field lines. This gauge is conveniently imposed directly on the basis functions, introducing the tree-cotree decomposition of the mesh and eliminating the degrees of freedom associated to the tree edges as described in Section 4. Condition $\mathbf{J}_k \cdot \hat{\mathbf{n}} = 0$ on ∂V_c can also be imposed on the shape functions using the approach described in Section 5. The shape functions are therefore derived from the n basis functions for the gauged vector potential:

$$\mathbf{J}_k = \nabla \times \mathbf{N}_k \qquad \text{(A.2.2)}$$

Matrix definition

In this way, Galerkin's method applied to (A.1.15) gives:

$$\underline{\underline{L}}\frac{d\underline{I}}{dt}+\underline{\underline{R}}\,\underline{I}=\underline{V} \tag{A.2.3}$$

where $\underline{I}$ is the column vector made of the I_k s and

$$\left(\underline{V}\right)_i \hat{=} -\int_{V_c}\nabla\times\mathbf{N}_i(\mathbf{r})\cdot\frac{\partial}{\partial t}\mathbf{A}_s(\mathbf{r},t)d\mathbf{r} \tag{A.2.4}$$

$$\left(\underline{\underline{R}}\right)_{ij} \hat{=} \int_{V_c}\nabla\times\mathbf{N}_i(\mathbf{r})\cdot\sigma^{-1}\nabla\times\mathbf{N}_j(\mathbf{r})d\mathbf{r} \tag{A.2.5}$$

$$\left(\underline{\underline{L}}\right)_{ij} \hat{=} \frac{\mu_0}{4\pi}\int_{V_c}\int_{V_v}\frac{\nabla\times\mathbf{N}_i(\mathbf{r})\cdot\nabla\times\mathbf{N}_j(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|}d\mathbf{r}'\,d\mathbf{r}\,. \tag{A.2.6}$$

Nodal shape functions for scalar fields

The nodal function N_k associated to the k-th node is continuous, piecewise polynomial with:

$$N_k(\mathbf{r}_k) = 1$$

$$N_k(\mathbf{r}_m) = 0, \quad k \neq m$$

$$\sum_{j=1,NP} N_j(\mathbf{r}) = 1, \quad \mathbf{r} \in V_d$$

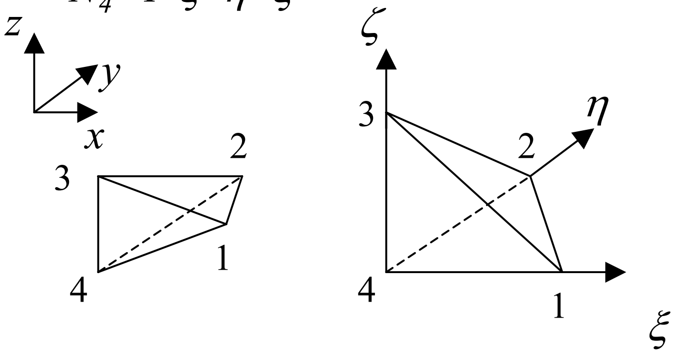
$$x = \sum_{k \in I_I} N_k(\xi,\eta,\zeta)\, x_k$$

$$y = \sum_{k \in I_I} N_k(\xi,\eta,\zeta)\, y_k$$

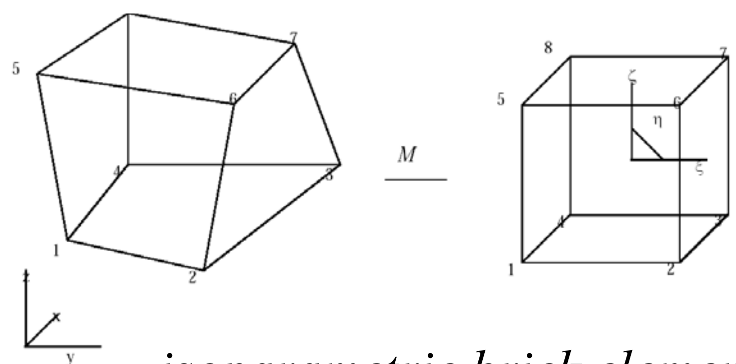
$$z = \sum_{k \in I_I} N_k(\xi,\eta,\zeta)\, z_k$$

$N_1=\xi$, $N_2=\eta$, $N_3=\zeta$

$N_4=1-\xi-\eta-\zeta$



$$N_k(\xi,\eta,\zeta) = \frac{1}{8}(1+\xi\xi_k)(1+\eta\eta_k)(1+\zeta\zeta_k)$$



isoparametric tetrahedron

isoparametric brick element

Edge element based shape functions for vector fields

- The line integral of $\mathbf{N}_e$ along the edge e (from node i to node j) is one:

$$\int_{\{i,j\}} \mathbf{N}_e \cdot d\mathbf{l} = 1 \qquad \text{(A.3.8)}$$

- The line integral of $\mathbf{N}_e$ is zero along any other edge $\{l,k\} \neq \{i,j\}$ and $\{k,l\} \neq \{i,j\}$:

$$\int_{\{l,k\}} \mathbf{N}_e \cdot d\mathbf{l} = 0 \qquad \text{(A.3.9)}$$

- The tangential components of $\mathbf{N}_e$ are continuous across the face of adjacent elements since the scalar functions N_k are continuous.
- The normal components of $\mathbf{N}_e$ are not necessarily continuous.
- $\mathbf{N}_e$ and ∇N_k belong to the same functional space: the gradient of a nodal shape function is given by a linear combination of the edge shape functions having in common that node:

$$\nabla N_k = \sum_{e=1,E} G_{ek} \mathbf{N}_e \qquad \text{(A.3.10)}$$

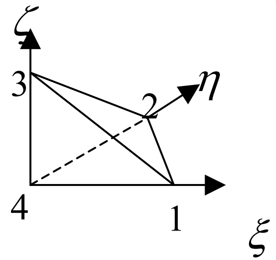
where E is the number of edges of the mesh and $\underline{\underline{\mathbf{G}}}$ is the $E \times NP$ incidence matrix defined by:

$$G_{em} = \begin{cases} +1 \text{ if } e = \{i,j\} \text{ and } m = j \\ -1 \text{ if } e = \{i,j\} \text{ and } m = i \\ 0 \text{ otherwise} \end{cases}$$

In a tetrahedron

$e = \{1,2\} \Rightarrow$

$$\mathbf{N}_e = N_1 \nabla N_2 - N_2 \nabla N_1$$



Tree-cotree decomposition

$$\nabla \times \mathbf{T}_1 = \nabla \times \mathbf{T}_2 = \mathbf{J} \qquad \mathbf{T}_1 \cdot \mathbf{w} = \mathbf{T}_2 \cdot \mathbf{w} = 0, \qquad \text{in } V \tag{46}$$

If V is a simply connected region, then the difference can be expressed as the gradient of a scalar function ψ:

$$\mathbf{T}_1 - \mathbf{T}_2 = \nabla \psi, \qquad \text{in } V \tag{47}$$

Let us fix a reference point P_o and take an arbitrary point P in V. If there always exists a field line γ of $\mathbf{w}$ that connects P to the reference point P_o, we get

$$\psi(P) - \psi(P_o) = \int_{P_o \gamma P} \nabla \psi \cdot \mathbf{t}\, dl = 0 \tag{48}$$

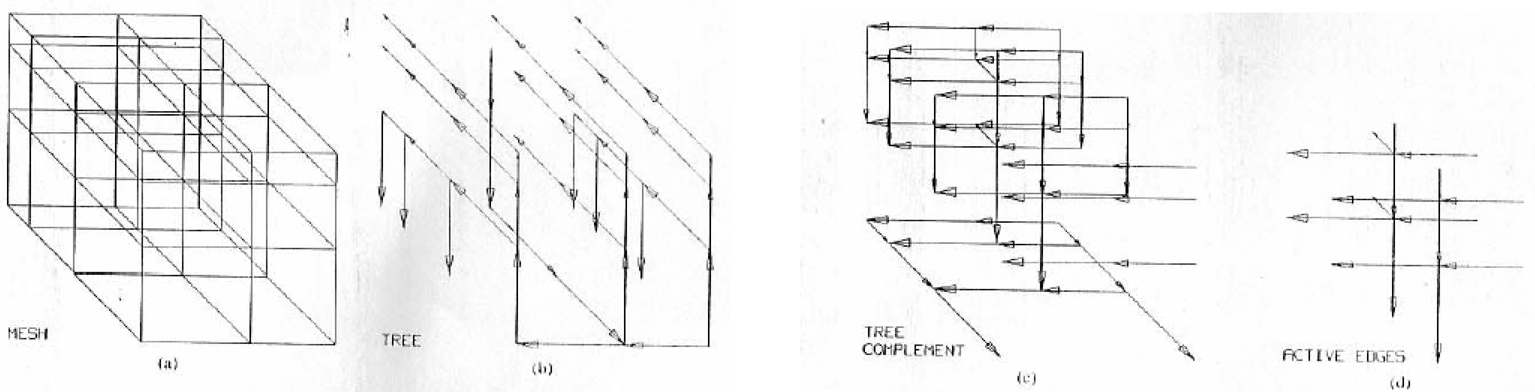
since $\mathbf{t}$, the unit tangent vector to γ, is parallel to $\mathbf{w}$ and $\nabla \psi \cdot \mathbf{w} = (\mathbf{T}_1 - \mathbf{T}_2) \cdot \mathbf{w} = 0$. Therefore, ψ is everywhere equal to its arbitrarily fixed value in P_o, hence $\mathbf{T}_1 - \mathbf{T}_2 = \nabla \psi = \mathbf{0}$, which proves the uniqueness of $\mathbf{T}$.

Let us refer to the graph formed by nodes and edges of the edge elements mesh. As shown by (75), the flux of $\mathbf{J} = \nabla \times \mathbf{T}$ across any elementary face is related to the circulation of the vector potential along the edges identifying the face. We decompose the graph in an arbitrary tree, formed by $P - 1$ edges, and the cotree, formed by the residual $E - P + 1$ edges. According to the fundamentals of the circuit theory, each edge of the cotree closes a single independent loop with the edges of the tree, and the circulation along any other loop can be obtained by linear combination of the circulations along the $E - P + 1$ independent loops associated with the edges of the cotree. Henceforth, and degrees of freedom corresponding to the edges of the tree can be eliminated, leaving a reduced set of coefficients

Boundary conditions for simply connected regions

be known. Obviously, to have Ω continuous at the interfaces between V_c, V_o and $V - V_c - V_o$, we should impose $\mathbf{T} \times \mathbf{n} = \mathbf{0}$ on the boundary of $V - V_c - V_o$. This is straightforward if $V - V_c - V_o$ is simply connected. In fact, in this case, we can use the same technique utilized to impose the boundary condition $\mathbf{B} \cdot \mathbf{n} = 0$ in the $\mathbf{A}$, φ formulation. Namely, we form the tree in such a way that a subset of it, the boundary tree, is a tree for the graph formed by nodes and edges lying on the boundary of $V - V_c - V_o$. In this way the degree of freedom associated with each boundary edge of the cotree is the total current crossing a surface lying on the boundary (related to the loop closed by the tree and that boundary edge), which must be zero. Henceforth, the condition is met by simply eliminating the degrees of freedom associated with the boundary edges of the cotree. In this way,

the basis functions $\mathbf{T}_e$ for the gauged potential $\mathbf{T}$ are given by the edge element shape functions $\mathbf{N}_e$ associated with the n_{ei} active edges.



Nonlinear eddy current problem

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}), \qquad \mathbf{M} = \mathcal{G}(\mathbf{B}), \qquad \text{in } V_1$$

weighted residual approach:

$$\int_{V_c} \mathbf{W} \cdot (\eta \mathbf{J} - \mathbf{E})\, dV = 0, \qquad \forall\, \mathbf{W} \in S$$

$$\int_{V_1} \mathbf{W}_M \cdot [\mathbf{M} - \mathcal{G}(\mathbf{B})]\, dV = 0, \qquad \forall\, \mathbf{W}_M \in \mathbf{L}^2(V_1)$$

$$\mathbf{J} \in S$$

$$\mathbf{M} \in \mathbf{L}^2(V_1)$$

- Convergence guaranteed for suitable Picard iterations (for monotonic $\mathcal{G}$)
- Further problems: hysteresis, motion, …

$$\begin{aligned}\mathbf{B}(\mathbf{x}, t) = \mathbf{B}_s(\mathbf{x}, t) &+ \frac{\mu_0}{4\pi}\int_{V_c} \frac{\mathbf{J}(\mathbf{x}', t) \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}\, dV' \\ &+ \mu_0 \mathbf{M} - \frac{\mu_0}{4\pi}\int_{V_1} \nabla' \cdot \mathbf{M}(\mathbf{x}', t) \frac{(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}\, dV' \\ &+ \frac{\mu_0}{4\pi}\int_{\partial V_1} \mathbf{M}(\mathbf{x}', t) \cdot \mathbf{n}' \frac{(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}\, dS'\end{aligned}$$

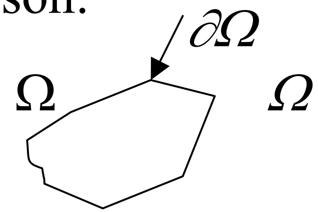
$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi}\int_{V_c} \frac{\mathbf{J}(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|}\, dV' + \frac{\mu_0}{4\pi}\int_{V_1} \frac{\mathbf{M}(\mathbf{x}', t) \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}\, dV' + \mathbf{A}_s(\mathbf{x}, t)$$

Formulazioni integrali e differenziali: *confronto*

- *Differenziali:*
  - *Discretizzazione anche dei domini omogenei (compresi aria e vuoto)*
  - *Problemi nel trattamento di domini non limitati*
  - *Campi generalmente continui e solenoidali solo in forma debole*
  - *Matrici sparse e calcolate rapidamente*
  - *Termini da integrare sufficientemente regolari*
  - *Trattamento automatico di mezzi non lineari, disomogenei ed anisotropi*
- *Integrali:*
  - *Limitato costo di discretizzazione*
  - *Condizioni di regolarità all'infinito tenute automaticamente in conto*
  - *Campi magnetici in genere continui e solenoidali*
  - *Matrici piene calcolate tramite integrali doppi di volume*
  - *Singolarità*
  - *Problemi nel trattamento di mezzi non lineari, disomogenei ed anisotropi*

Cenni sui "boundary elements" (1)

- applicazione al problema di Poisson:
  - in dominio non limitato Ω
  - con Dirichlet BC:



$$\begin{cases} \nabla^2\varphi = q \quad in\ \Omega \\ \varphi = f \quad su\ \partial\Omega \\ \varphi\ regolare\ all'\ infinito \end{cases}$$

- Formule di Green:

$$\int_\Omega (\psi\nabla^2\varphi - \varphi\nabla^2\psi)\,d\Omega = \int_{\partial\Omega} (\psi\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial\psi}{\partial n})\,d\Sigma$$

- Equazioni integrali al contorno utilizzando la soluzione fondamentale *F*

$$\int_\Omega Fq\,d\Omega - \alpha\varphi = \int_{\partial\Omega} (F\frac{\partial\varphi}{\partial n} - f\frac{\partial F}{\partial n})\,d\Sigma$$

$$\begin{cases} \nabla^2 F = \delta \quad in\ R^n \\ \varphi\ regolare\ all'\ infinito \end{cases} \qquad \alpha = \begin{cases} 1 & in\ \Omega \\ 1/2 & su\ \partial\Omega\ (punti\ regolari) \\ \dots \end{cases}$$

$$F=1/4\pi|x-x'|\ in\ R^3$$

Cenni sui “boundary elements” (2)

- Applicazione del metodo dei residui pesati:

$$\int_{\partial\Omega} w'\left(\int_{\Omega} Fq\,d\Omega - \alpha\varphi\right) d\Sigma' = \int_{\partial\Omega}\int_{\partial\Omega} w'\left(F\frac{\partial\varphi}{\partial n} - f\frac{\partial F}{\partial n}\right) d\Sigma\, d\Sigma'$$

- Scelta delle incognite: *$\partial\varphi/\partial n$, σ (densità di sorgenti superfic. $\delta[\partial\varphi/\partial n]$ nel problema ausiliario interno+esterno); φ dove non c’è Dirichlet BC*
- Funzioni di base e di peso *(ammissibili anche costanti a tratti)*
- Confronto con altri metodi
  - Singolarità
  - Discretizzazione richiesta solo sul contorno
  - Necessità di disporre della soluzione fondamentale
- Accoppiamento con altri metodi (es. FEM-BEM)