Lezione n. 9

Stima dell’errore.

Minimizzazione dell’errore costitutivo.

Stima dell'errore

- Per valutare l'accuratezza della soluzione ottenuta
- Per migliorare la soluzione ottenuta ove necessario (mesh refinement)
- Stime *a priori* ed *a posteriori*
- Tecniche principali:
  - Espansione locale
  - Correlazione con le discontinuità tra elementi adiacenti
  - Errore costitutivo

Constitutive error approach (1)

THE DC PROBLEM

Canonical field equations (local form of Kirchhoff's laws)

$\nabla\times\mathbf{E}=\mathbf{0}$ in V

$\nabla\cdot\mathbf{J}=0$ in V

Constitutive equation (local form of Ohm's law)

e.g. $\mathbf{J}=\sigma(\mathbf{E}+\mathbf{E}_i)$ in V

Boundary condition

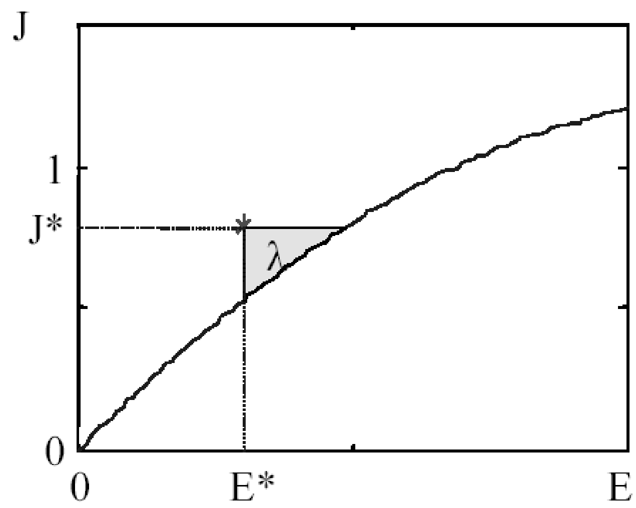
e.g. $\mathbf{J}\cdot\mathbf{n}=0$ on ∂V

V conducting domain, $\mathbf{E}$ electric field, $\mathbf{J}$ current density, $\mathbf{E}_i$ impressed electric field σ electrical conductivity, j and $\mathbf{e}$ prescribed boundary conditions

Constitutive error approach (2)

ERROR BASED APPROACH

- *introduction of* **T** *and* ϕ *potentials, to satisfy automatically the canonic equations* $\nabla\times\mathbf{E}=\mathbf{0}$ *and* $\nabla\cdot\mathbf{J}=0$ *by imposing* $\mathbf{J}=\nabla\times\mathbf{T}$ *and* $\mathbf{E}=-\nabla\phi$ *;*
- *definition of local error density* λ*:*
  - *linear passive case* $\rightarrow \lambda = (\mathbf{J}-\sigma\mathbf{E})^2/\,2\sigma = [\nabla\times\mathbf{T}+\sigma\nabla\phi)]^2/\,2\sigma$;
  - *general non-linear case:*



Constitutive error approach (3)

Error based formulation of DC problem

$$\nabla\times\mathbf{E}=\mathbf{0} \qquad\qquad \nabla\cdot\mathbf{J}=0$$

$$\mathbf{E}=-\nabla\phi \qquad\qquad \mathbf{J}=\nabla\times\mathbf{T}$$

$$+\ \ \phi=\phi^* \quad \text{on } \partial V_\phi \qquad\qquad +\ \mathbf{J}\cdot\mathbf{n}=J^* \text{ on } \partial Vj$$

$$\mathbf{J} = \sigma\,(\,\mathbf{E} + \mathbf{E}_i)$$

$$\Lambda(T,\phi) = \int_V \lambda(T,\phi)\,dV = \int_V [\mathbf{J}-\sigma(\mathbf{E}+\mathbf{E}_i)]^2/\sigma\,dV$$

global error functional to be minimized

Constitutive error approach (4)

Numerical FEM formulation of DC problem

$$\min \Lambda(\mathbf{T}, \phi)$$

$$\min \int_V [\mathbf{J}-\sigma(\mathbf{E}+\mathbf{E}_i)]^2/\sigma \, dV = \min \int_V [\nabla\times\mathbf{T}-\sigma(-\nabla\phi+\mathbf{E}_i)]^2/\sigma \, dV$$

$\Lambda(\mathbf{T},\phi) > 0 \qquad \forall (\mathbf{T},\phi) \neq (\mathbf{T}_0,\phi_0)$

$\Lambda(\mathbf{T}_0,\phi_0) = 0 \qquad$ where $(\mathbf{T}_0,\phi_0)$ is the actual solution

$\mathbf{E} = -\sum c_k \nabla\lambda_k(\mathbf{x}) \qquad$ (λ_k nodal isoparametric shape functions)

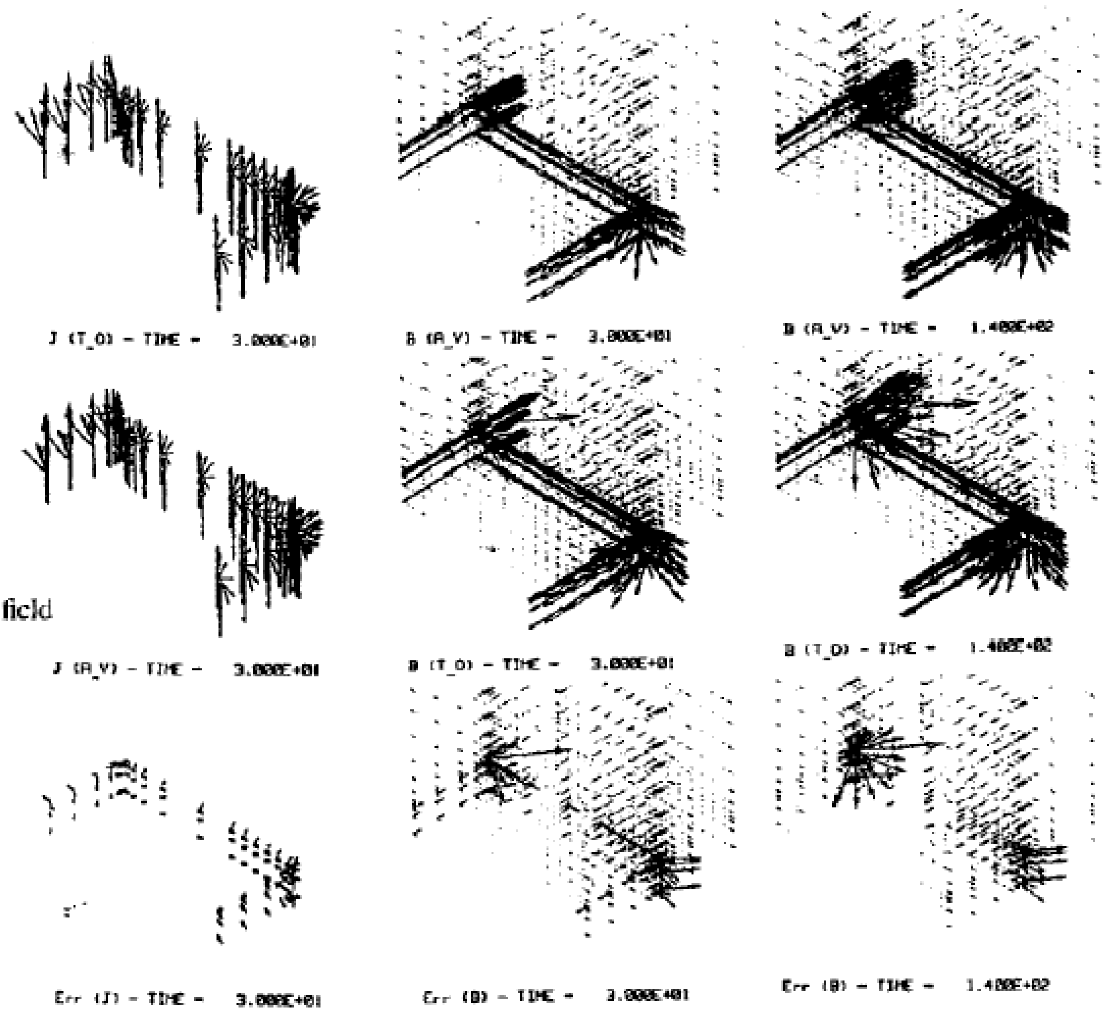
$\mathbf{J} = \sum I_k \nabla\times\mathbf{N}_k(\mathbf{x}) \qquad$ ($\mathbf{N}_k$ edge-element based shape functions)

Λ becomes a quadratic form in the coefficients c_k and I_k $\rightarrow$
$\rightarrow$ sparse linear system for linear constitutive equations

Constitutive error approach (5)

Error estimation and criteria for mesh refinement

FIGURE 28. TEAM workshop Problem 10. Results obtained by the differential formulations with the fine mesh at the times $t = 30$ ms and $t = 140$ ms. Here $B(A\_V)$ is the flux density $\mathbf{B} = \nabla \times \mathbf{A}$; $B(T\_O)$ is the flux density $\mathcal{B}(\mathbf{H}) = \mathcal{B}(\mathbf{T} - \nabla\Omega + \mathbf{T}_s)$; $\mathrm{Err}(B)$ is the error field $\mathbf{B} - \mathcal{B}(\mathbf{H})$; $J(T\_O)$ is the current density $\mathbf{J} = \nabla \times (\mathbf{T} + \mathbf{T}_s)$; $J(A\_V)$ is the current density $\mathbf{J}_s + \sigma\mathbf{E} = \mathbf{J}_s - \sigma(\partial\mathbf{A}/\partial t + \nabla\partial\phi/\partial t)$; $\mathrm{Err}(J)$ is the error field $\mathbf{J} - \mathbf{J}_s - \sigma\mathbf{E}$.



Constitutive error approach (6)

Error estimation and criteria for mesh refinement (cont'd)

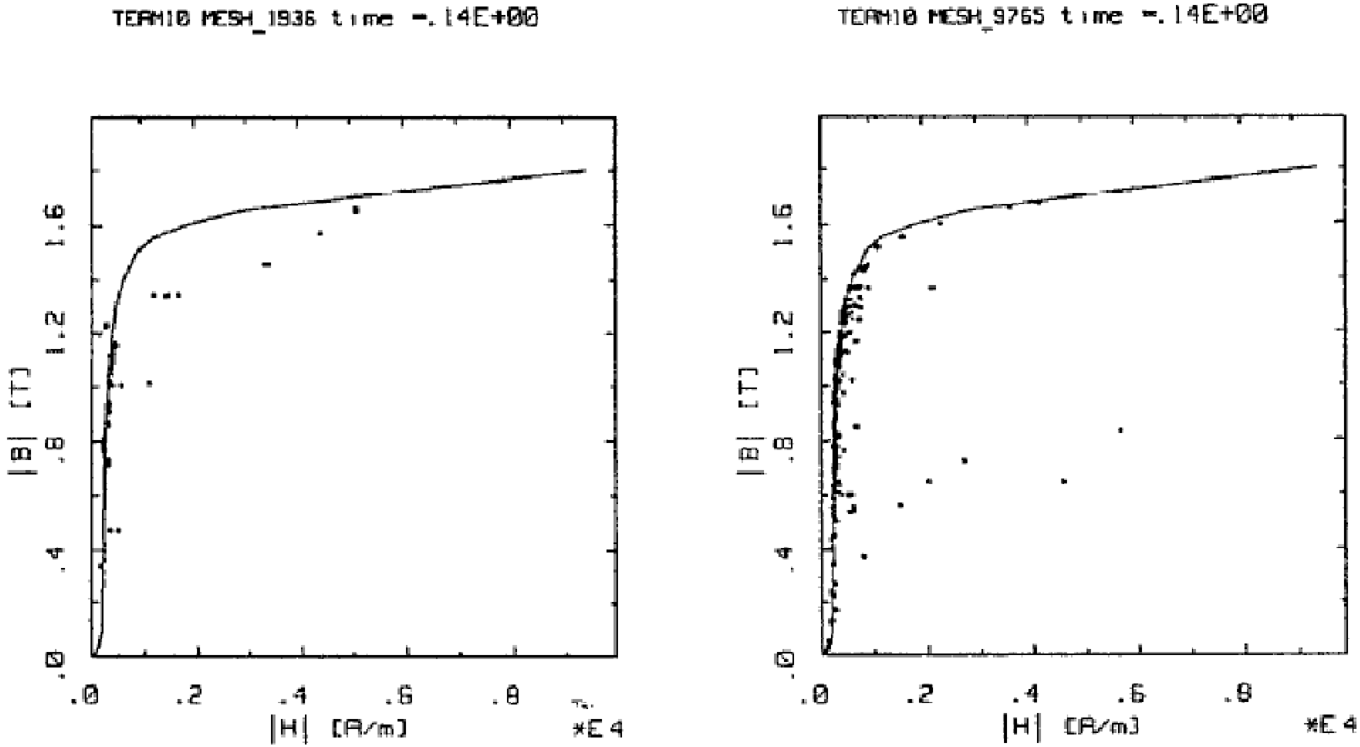


FIGURE 29. TEAM workshop Problem 10. Results obtained in the iron by the differential formulation with coarse and fine meshes at the time $t = 140$ ms. B-H plots (pairs at the element centroids) with $B = |\nabla \times \mathbf{A}|$ and $H = |\mathbf{T} - \nabla\Omega + \mathbf{T}_s|$. (Albanese and Rubinacci, 1994.)

Constitutive error approach (7)

Main features of the costitutive error approach

- Straightforward formulation of most problems in engineering and applied sciences (including coupled problems) characterized by (usually linear) field equations and (linear or nonlinear) constitutive equations
- Straightforward extension to transient or steady-state problems
- No need of weak formulations
- Automatical treatment of material discontinuities
- Straightforward (qualitative) error estimation and criteria for mesh refinement
- Global and local (quantitative) error bounds
- Double number of unknowns

Global and Local Bounds (1)

Splitting

Under suitable (e.g., homogeneous) B.C. the global error **splits** into

$$\Lambda(\mathbf{T},\phi) = \Xi(\mathbf{T}) + \vartheta(\phi) \geq \Xi(\mathbf{T}_0) + \vartheta(\phi_0) = 0$$

e.g., for homogeneous B.C.:

$$-\,\Xi = -\int_V (\nabla\times\mathbf{T})^2/\sigma \, dV + 2 \int_V (\nabla\times\mathbf{T})\cdot\mathbf{E}_i \, dV$$

$$\vartheta = \int_V \sigma(-\nabla\phi + \mathbf{E}_i)^2 \, dV$$

splitting made possible by virtual power principle:

$$\int_V \mathbf{J}\cdot\mathbf{E} \, dV = -\int_V (\nabla\times\mathbf{T})\cdot \nabla\phi \, dV = 0$$

Global and Local Bounds (2)

Global Bounds (linear or nonlinear stationary problems)

The *actual* ***power*** in the whole domain V is

$$P_0 = -\,\Xi(\mathbf{T}_0) = \vartheta(\phi_0)$$

$$\Xi(\mathbf{T}_0) + \vartheta(\phi) \geq \Xi(\mathbf{T}_0) + \vartheta(\phi_0) \longrightarrow P_U = \vartheta(\phi) \geq \vartheta(\phi_0)$$

$$\Xi(\mathbf{T}) + \vartheta(\phi_0) \geq \Xi(\mathbf{T}_0) + \vartheta(\phi_0) \longrightarrow P_L = \Xi(\mathbf{T}) \geq \Xi(\mathbf{T}_0)$$

Hence, the bounds:

$$P_L(\mathbf{T}) = -\Xi\,(\mathbf{T}) \leq -\Xi\,(\mathbf{T}_0) = P_0 = \vartheta(\phi_0) \leq \vartheta(\phi) = P_U(\phi)$$

Global and Local Bounds (3)

Local Bounds (linear stationary problems)

Based on well-known expression for mutual energy, co-energy, power, resistance, inductance, …

$L = L_1 + L_2 + 2\ M$ (self-inductance of the series of two circuits)

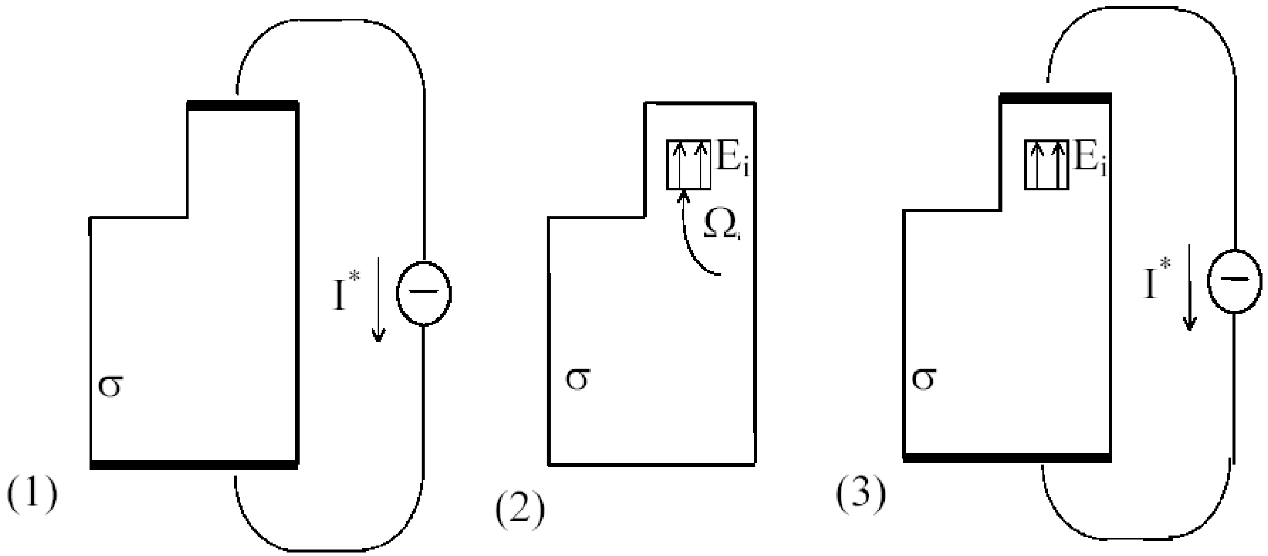
Upper and lower bounds available for $L, L_1, L_2 \rightarrow$ hence for M

Mutual inductance between a test coil and the source currents = average field linked with the test coil $\times$ cross-section

Auxiliary problem with artificial source (can shrink to a point)

Global and Local Bounds (4)

Real problem *auxiliary problem* *superposition*



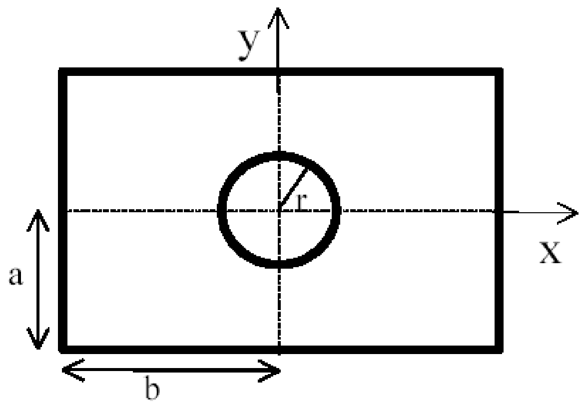
∂V_{j0} : boundary part where $\mathbf{J}\cdot\mathbf{n} = J^* = 0$;

∂V_{j1} : boundary part where $\mathbf{J}\cdot\mathbf{n} = J^* \neq 0$;

By solving three problems, bounds can be established for the average value along one direction of the current density in the Ω_i region.

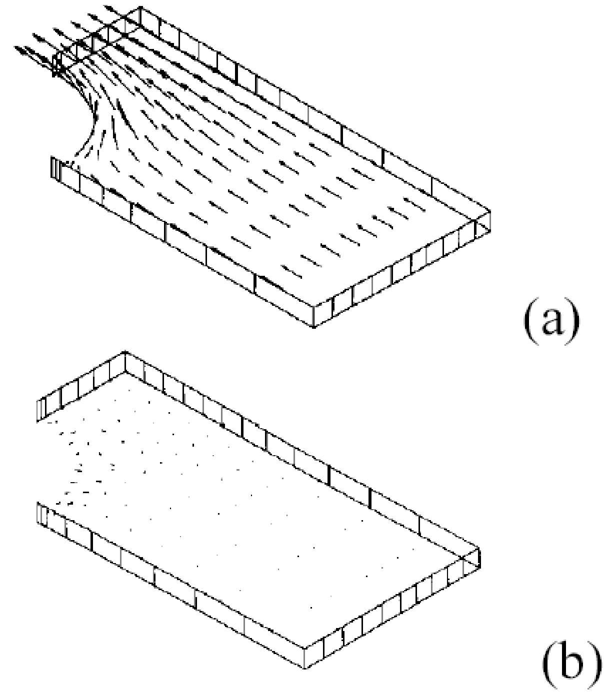
Global and Local Bounds (5)

Resistance of a plate with a 2D array of circular holes (approximate analytical Clausius-Mossotti expression acceptable for small r/a values)



Elementary cell of a plate with cylindrical holes of radius r (b/a=2). Equivalent conductivity tensor σ (S/m) (σ_M=1 MS/m), compared to σ_{CM}

r/a=0.25	r/a=0.50	r/a=0.75
σ_{xx}=950861±169	σ_{xx}=802026±1354	σ_{xx}=547053±380
σ_{yy}=953351±151	σ_{yy}=837201±437	σ_{yy}=697105±681
σ_{CM}=963631	σ_{CM}=859628	σ_{CM}=701616



Numerical analysis of a plate with circular holes: (a) current density given by the **T** solution; (b) error estimate **J**-σ**E**

Applicazione del metodo dell'errore costitutivo ai circuiti

A TITOLO ESEMPLIFICATIVO

- Equazioni di campo ↔ Equazioni di Kirchhoff
- Equazioni costitutive ↔ Caratteristiche dei bipoli
- Potenziale scalare elettrico ↔ Potenziali di nodo
- Potenziale vettore elettrico ↔ Correnti di maglia
- Proprietà