Metodi e tecniche di ottimizzazione innovative per applicazioni elettromagnetiche

Ottimizzazione multiobiettivo

Parte 1 Ottimo di Pareto

Maurizio Repetto

Politecnico di Torino, Dip. Ingegneria Elettrica Industriale

Contents

- Definition of Vector Optimization problem
- Pareto Optimality
- Vector Optimization Methods
  - Pareto based methods
  - Aggregation methods
    - Weighting of objectives
    - Constrained approach
    - Goal
    - Fuzzy

Vector Optimization problem

- Practical optimization problems often requires to minimize/maximize several objectives at the same time
- These problems are usually referred as Vector Optimization Problems (*VOP*) or Multiobjective Optimization Problems (*MOP*) and are defined as:

maximize { O }
{ O }= { O_j(**X**)} j=1,..,M

subject to $g_i(\mathbf{X}) \leq 0, i = 1, ...P$ { O_j(**X**)} j-th objective **X** degrees of freedom vector

VOP development (1/2)

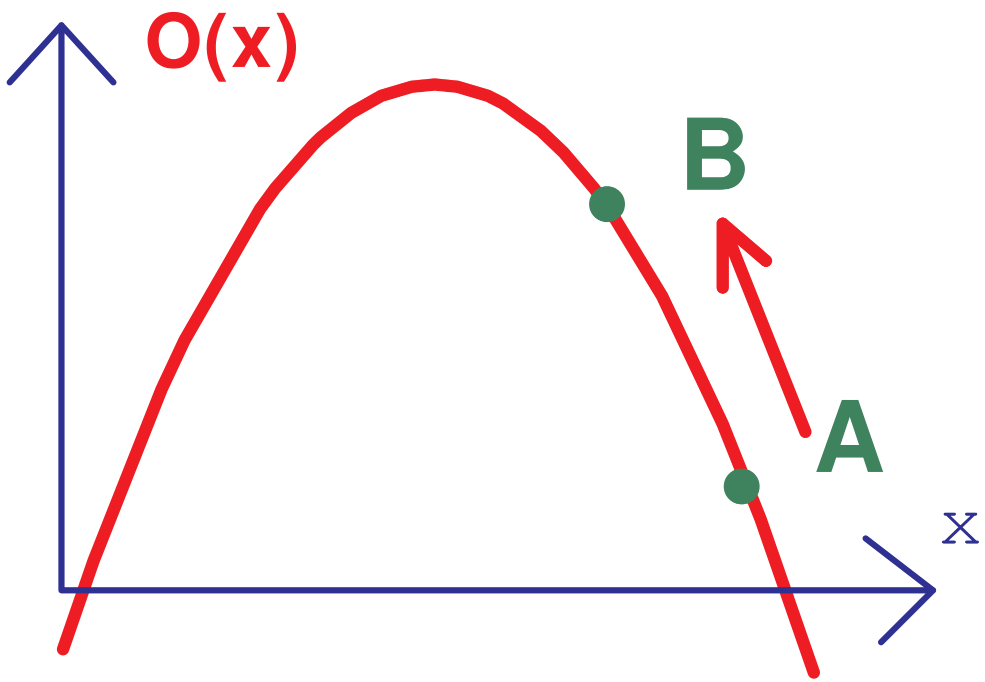
- VOP has always attracted interest by many sceintists trying to define the best choice among a bundle of alternatives
- Initially this study was dedicated to economy and the firsy approach to VOP is attributed to Jeremy Bentham who in 1789 published a theory of "utility"
- if any action has a numerical value, it is possible to define in a univoque way the best alternative between two(cardinal utility function)
- unfortunately it is not always univoque how to map an "action" to a real number

VOP development (2/2)

- After the work of Bentham, in the beginning of 1900 Pareto defined in a mathematical way an optimality criterion which has been extensively used in welfare economy and in operation research
- The study of this subject has later evolved in many branches like game theory, programming and planning in economy etc.
- in the field of engineering, the study was mainly devoted to the analysis of risky enviromental and control
- recently it has applied also to design problems trying to find automatically the best choice in a set of possible configurations.

Pareto optimality (1/3)

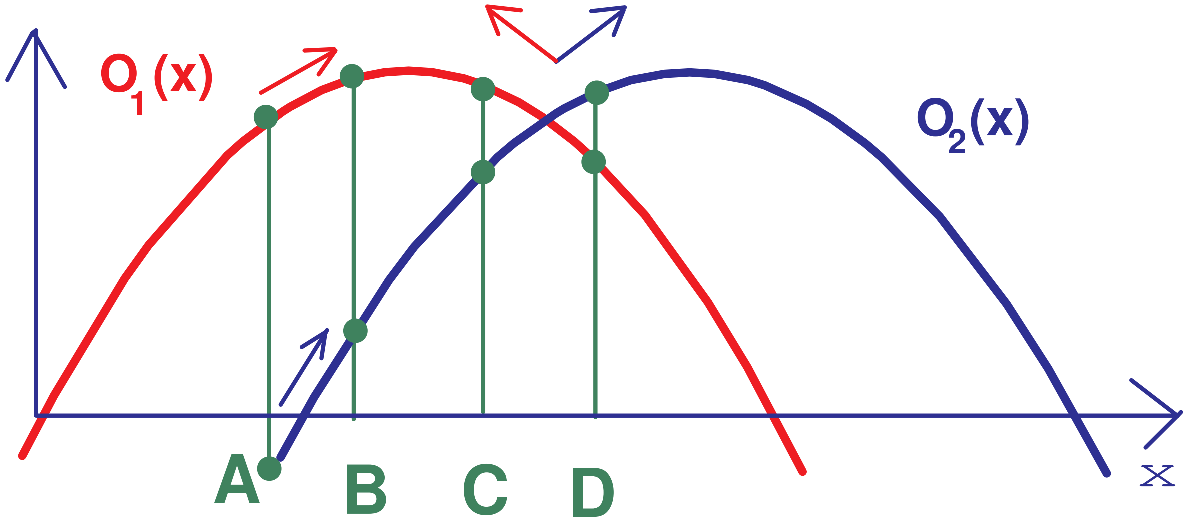
- In scalar optimization problems the definition of optimal point is univoque: in maximization a greater value of objective is preferred and any move "should" go in this direction



B is better than A

Pareto optimality (2/3)

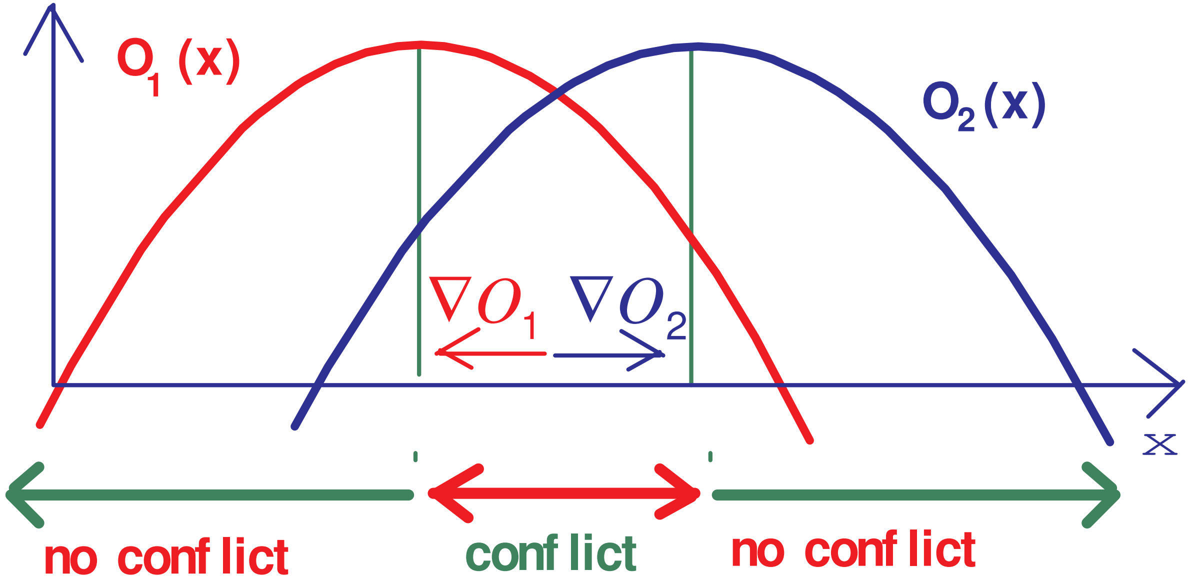
- In vector optimization problem comparison is more difficult, if O_1 and O_2 must be maximized



B is better than A	C is not comparable with D
$O_1(A) < O_1(B)$	$O_1(C) < O_1(D)$
and	but
$O_2(A) < O_2(B)$	$O_2(C) < O_2(D)$

Pareto optimality (3/3)

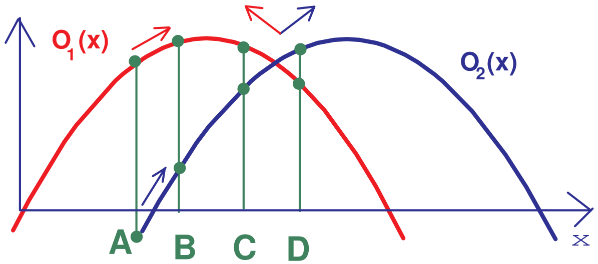
- The domain of the problem can be divided in three sub-domains:



- gradients of the functions have opposite directions in the conflict zone

Pareto front (1/3)

- The conflict zone is in general a sub set of the domain and is called Pareto front
- Configurations on the Pareto front are not comparable because
  an improvement in one causes a degradation in the other

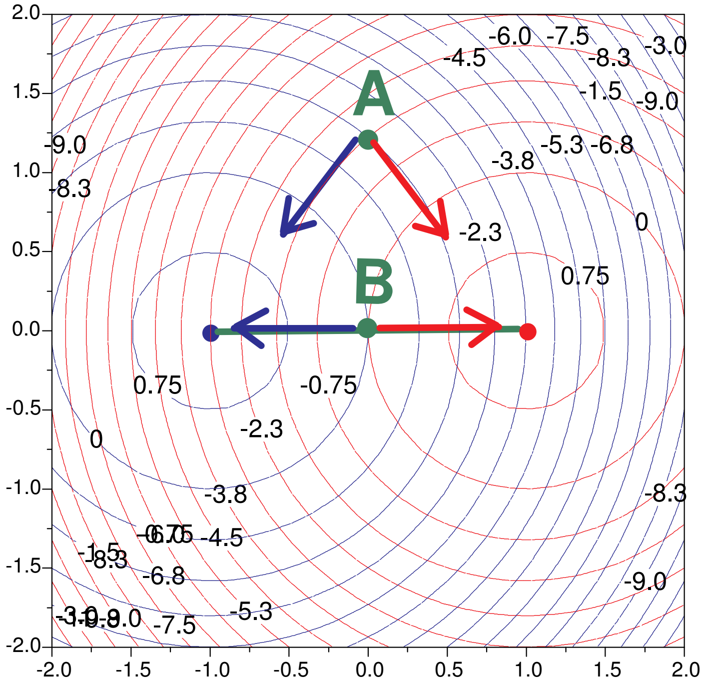


A is inferior to B
or
A is dominated by B

C is not comparable with D
or
C and D are not inferior

Pareto front (2/3)

- Pareto front can be located by looking for noninferior solutions or by looking at the gradient of the objectives

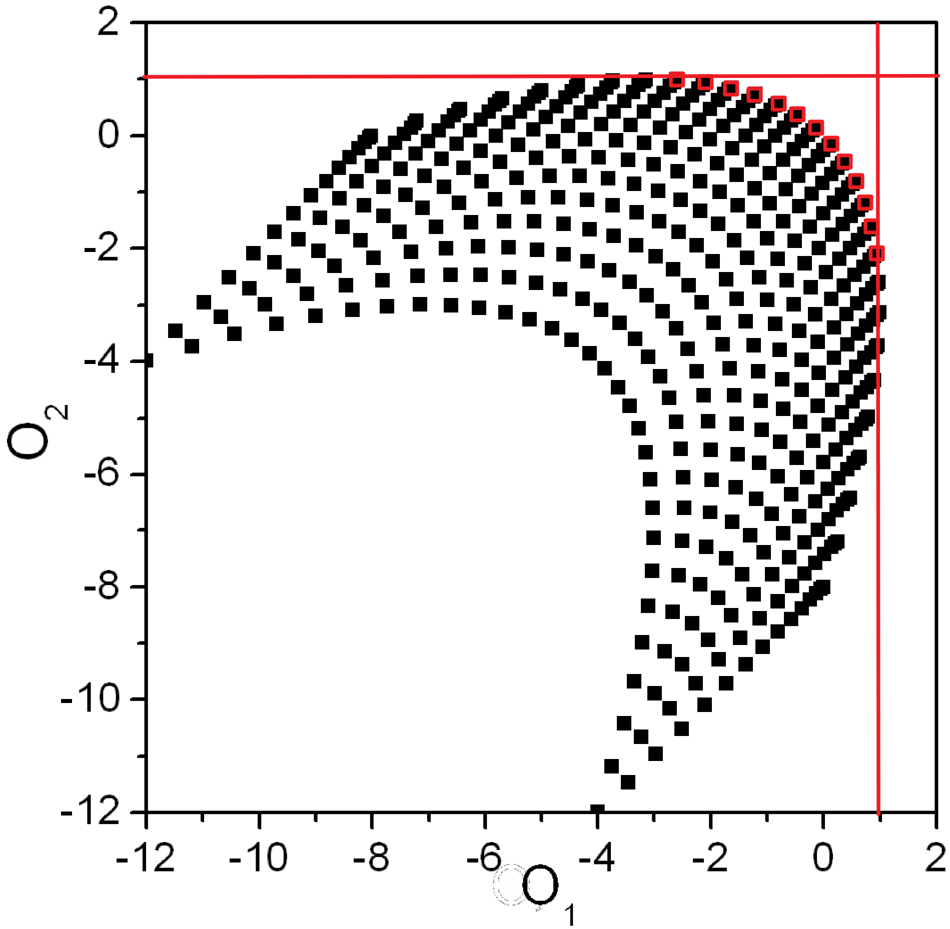


both objectives can be improved starting by point A moving along the gradient directions

- objectives in point B are in conflict and thus B lies on the Pareto front all points on this line are noninferior or nondominated solutions

Pareto front (3/3)

- Graphical representations of the problem can be useful also in the objective functions space



$$max\{O_1, O_2\}$$
$$O_1 = (x-1)^2 + y^2$$
$$O_2 = (x+1)^2 + y^2$$

Non inferiority-Dominance

- One point X_P is a Pareto optimal point if there exist no other solution X for which holds:

$$O_i(X) \geq O_j(X_P), i = 1, ...M, i \not k O_k(X) > O_k(X_P)$$

- In comparing two solutions X_A and X_B A dominates B, or B is inferior to A, if:

$$O_i(X_A) \geq O_i(X_B), i = 1, ...M, i$$

  and there is at least one k

$$O_k(X_A) > O_k(X_B)$$

- If none of the two solutions dominates over the other, the two solutions are nondominated or noninferior
- Dominance or noninferiority are thus the operative concepts to look for Pareto optimal solutions

VOP key concepts

- Conflict: not all the objectives can be optimized at the same time because increasing one objective leads to the deterioration of the others
- Pareto front: set of all configurations which exhibit conflict among objectives
- Tradeoff: is the amount of one objective that must be sacrified to obtain an increase in the others
- Best compromise solution: the solution preferred by the user which represents a bargain between different instances
- Indifference curve: a set of equally "best" solutions among which the user cannot choose

Kuhn-Tucker conditions

- Kuhn-Tucker (1951) developed a set of optimality conditions for scalar constrained optimization problems. They also stated conditions for noninferiority in VOP.
- given a vector maximization problem, if a solution X_P is noninferior then there exist w_k positive multipliers k=1,..,M so that

$$\sum_{k=1}^{M} w_k \nabla O_i(\mathbf{X}_P) = 0 \quad w_k > 0,\ k = 1, ..., M$$

- the condition is necessary and can become sufficient if some hypothesis can be stated on the VOP domain (convexity) and on functions O (concavity).

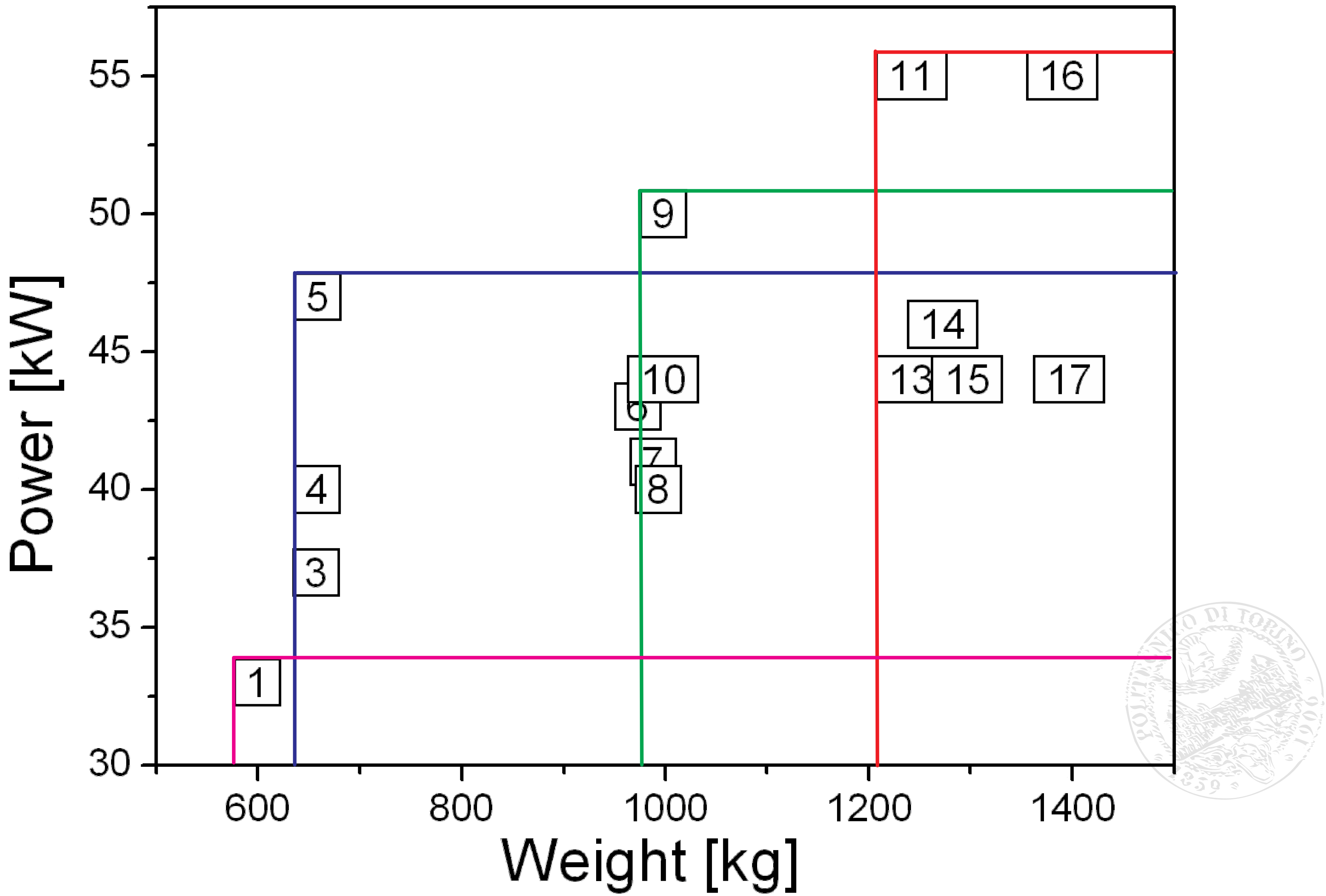
Example of Pareto Optimality (1/3)

- Choice of car among a set taking into account two criteria
  - weight to be minimal
  - power to be maximal
- not necessarily the choice will be univocal

Example of Pareto Optimality (2/3)

	Make of car	Weight [kg]	Power [kW]
1	smart	599	33
2	mitsubishi toppo	657	37
3	mitsubishi minica	657	37
4	mazda carol	658	40
5	honda zz turbo 4x4	659	47
6	opel agila	973	43
7	daihatsu cuore	989	41
8	suzuki alto	993	40
9	toyota yaris	998	50
10	nissan micra	998	44
11	mazda 121	1242	55
12	fiat punto	1242	44
13	lancia y	1242	44
14	mini	1273	46
15	ford ka	1297	44
16	audi a2	1390	55
17	skoda fabia	1397	44

Example of Pareto Optimality (3/3)



Strategies for VOP

- As it has been pointed out VOP show peculiarities that require special treatments and different approaches can be devised to this aim.
- Two main lines can be pointed out:
  - Pareto-based approaches, they look for the Pareto front without making any choice among the noninferior solutions (*impartiality*)
  - Aggregation approaches, by defining some preference criterion they combine the objectives into an higher scalar function which is dealt with by a scalar optimization method

Pareto based approach

- These methods have been proposed firstly by Goldberg in 1989 and are naturally linked to Evolutionary techniques or Genetic Algorithms (GA) because they need a "population" of solutions
- The basic idea is to define a fitness function related to a Pareto ranking where noninferior or nondominated solutions have a fitness greater than dominated solutions
- The outcome of the GA should be a population spread along the Pareto front
- Obviously the values of the single objectives are not taken into account so that after GA run a successive selection phase is needed

Pareto ranking

- A ranking of different solutions should fulfill the following requirements:
  - a assign better fitness values to noninferior solutions (*ND* set)
  - b assign a fitness value to dominated solutions (*DO* set) depending on their distance from the front in order to move the search in that direction
- Also this task can be approached in different ways:
  - Strength Pareto Evolutionary Approach
  - hierarchical selection

SPEA approach (1/6)

- The Strength Pareto Evolutionary Approach (*SPEA*) has been proposed in order to assign a minimal value of fitness to individuals on Pareto front
- the base of fitness computation is the boolean dominance matrix:

$$d_{i,j} = \begin{cases} 1 & \text{if } i \text{ is dominated by } j \\ 0 & \text{se } i \text{ is not dominated } j \end{cases} \quad d_{i,i} notused$$

- individuals which have all 0 on their row $\in ND$
- individuals which have at least a 1 on their row $\in DO$

SPEA approach (2/6)

- for individuals $\in ND$ fitness is computed as:

$$s_i = \frac{n}{N+1}$$

  where n is the number of individuals dominated by i, N is the total number of individuals and $f_i = s_i$

- for individuals $\in DO$ fitness is computed as:

$$f_j = 1 + \sum_{i, i \; dominate \; j} s_i$$

SPEA approach (3/6)

- example with two objectives to be minimized on 1D domain with 8 sampling points

$$O_1 = (x-1)^2 \quad O_2 = (x-2)^2$$

x	O_1	O_2
-2.972167	15.778112	24.722446
-1.275643	5.178552	10.729838
0.325297	0.455224	2.804631
0.478561	0.271899	2.314777
1.296701	0.088031	0.494630
1.374889	0.140542	0.390763
1.660726	0.436558	0.115107
2.151830	1.326711	0.023052

SPEA approach (4/6)

- the following dominance matrix can be computed:

$$D = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- $ND = \{x_5, x_6, x_7, x_8\}$. $DO = \{x_1, x_2, x_3, x_4\}$ and $N = 4$.

SPEA approach (5/6)

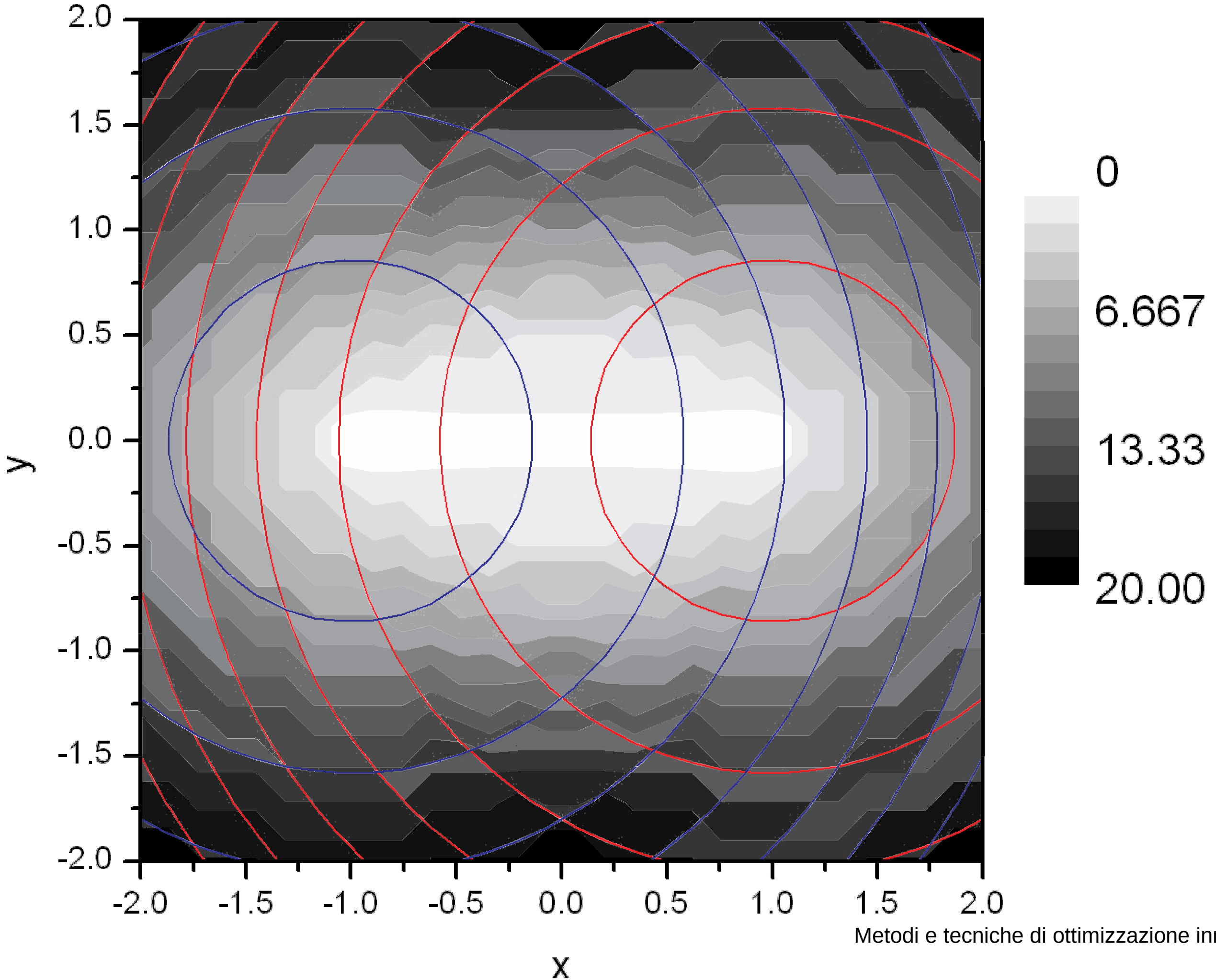
- the following SPEA fitness values can be computed:

$$\begin{array}{ll}
ND & DO \\
f_5 = \frac{4}{4+1} = 0.8 & f_1 = 1 + s_5 + s_6 + s_7 + s_8 = 3.6 \\
f_6 = \frac{4}{4+1} = 0.8 & f_2 = 1 + s_5 + s_6 + s_7 + s_8 = 3.6 \\
f_7 = \frac{3}{4+1} = 0.6 & f_3 = 1 + s_5 + s_6 + s_7 = 3.2 \\
f_8 = \frac{2}{4+1} = 0.4 & f_4 = 1 + s_5 + s_6 = 2.6
\end{array}$$

- an algorithm looking for a minimum is moving the search towards the Pareto front

SPEA approach (6/6)

- The SPEA approach can be applied to the two objectives 2D problem already presented where the Pareto front is known.



Pareto ranking

- Pareto techniques are efficient in finding the conflict area but do not give a precise indication of the optimum
- for instance in the previous example two configurations:
  $A \longrightarrow O_1 = 1, O_2 = -12$
  $B \longrightarrow O_1 = -12, O2 = 1$
- share the same fitness value because they are noninferior
- after the conclusion of the GA run, another phase has to begin which selects a best compromise solution
- In this phase the knowledge of the Pareto front can help a quantitative evaluation of the tradeoff among objectives